The Quantum Wave Packet of the Schrödinger's Equation for Continuous Quantum Measurements

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Abstract: In this paper we study the quantum wave packet of the Schrödinger's equation for continuous quantum measurements.

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1. Introduction

In this paper we will study the wave packet of a Schrödinger Continuous Measurements Equation proposed by Nassar [1], using the quantum mechanical formalism of de Broglie-Bohm.[2]

2. The Continuous Measurement Schrödinger's Equation

According to Nassar [1] a Schrödinger equation assuming continuous measurements is given by:

$$\mathrm{i}\,\hbar\,\frac{\partial\Psi(x,\ t)}{\partial t}\ =\ -\ \frac{\hbar^2}{2\ m}\ \frac{\partial^2\ \Psi(x,\ t)}{\partial x^2}\ +\ \Big[\ \tfrac{1}{2}\ m\ \Omega^2(t)\ x^2\ +\ \lambda\ x\ X(t)\ \Big]\ \Psi(x,\ t)\ -$$

$$-\ \frac{i\ \hbar}{4\ \tau}\ \left(\ \frac{[x\ -\ q(t)]^2}{\delta^2(t)}\ -\ 1\ \right)\ \Psi(x,\ t)\ , \qquad (2.1)$$

where $\Psi(x,\ t)$ is a wave function which describes a given system, $X(t)$ is the position of a classical particle submitted to a time dependent harmonic potential with frequency $\Omega(t)$, and τ and δ have dimensions of space and time, respectively, and $q(t)$ is average value $<\ x(t)\ >$.

Writing the wave function $\Psi(x,\ t)$ in the polar form defined by the Madelung-Bohm transformation [3,4] we obtain:

$$\Psi(x,\ t)\ =\ \phi(x,\ t)\ e^{i\ S(x,\ t)}\ , \qquad (2.2)$$

where $S(x\ ,t)$ is the classical action and $\phi(x,\ t)$ will be defined in what follows.

Calculating the derivatives, temporal and spatial, of (2.2), we get:

$$\frac{\partial\Psi}{\partial t}\ =\ e^{i\ S}\ \frac{\partial\phi}{\partial t}\ +\ i\ \phi\ e^{i\ S}\ \frac{\partial S}{\partial t}\ \ \rightarrow$$

$$\frac{\partial\Psi}{\partial t}\ =\ e^{i\ S}\ (\frac{\partial\phi}{\partial t}\ +\ i\ \phi\ \frac{\partial S}{\partial t})\ =\ (i\ \frac{\partial S}{\partial t}\ +\ \frac{1}{\phi}\ \frac{\partial\phi}{\partial t})\ \Psi\ , \qquad (2.3a,b)$$

$$\frac{\partial\Psi}{\partial x}\ =\ e^{i\ S}\ (\frac{\partial\phi}{\partial x}\ +\ i\ \phi\ \frac{\partial S}{\partial x})\ =\ (i\ \frac{\partial S}{\partial x}\ +\ \frac{1}{\phi}\ \frac{\partial\phi}{\partial x})\ \Psi\ , \qquad (2.3c,d)$$

$$\frac{\partial^2\Psi}{\partial x^2}\ =\ \frac{\partial}{\partial x}\ [(i\ \frac{\partial S}{\partial x}\ +\ \frac{1}{\phi}\ \frac{\partial\phi}{\partial x})\ \Psi]\ =$$

$$=\Psi\ [i\ \frac{\partial^2 S}{\partial x^2}\ +\ \frac{1}{\phi}\ \frac{\partial^2\phi}{\partial x^2}\ -\ \frac{1}{\phi^2}\ (\frac{\partial\phi}{\partial x})^2]\ +\ (i\ \frac{\partial S}{\partial x}\ +\ \frac{1}{\phi}\ \frac{\partial\phi}{\partial x})\ \frac{\partial\Psi}{\partial x}\ =$$

$$=\Psi\ [i\ \frac{\partial^2 S}{\partial x^2}\ +\ \frac{1}{\phi}\ \frac{\partial^2\phi}{\partial x^2}\ -\ \frac{1}{\phi^2}\ (\frac{\partial\phi}{\partial x})^2]\ +\ (i\ \frac{\partial S}{\partial x}\ +\ \frac{1}{\phi}\ \frac{\partial\phi}{\partial x})\ (i\ \frac{\partial S}{\partial x}\ +\ \frac{1}{\phi}\ \frac{\partial\phi}{\partial x})\ \Psi\ =$$

$$=\Psi\ [i\ \frac{\partial^2 S}{\partial x^2}\ +\ \frac{1}{\phi}\ \frac{\partial^2\phi}{\partial x^2}\ -\ \frac{1}{\phi^2}\ (\frac{\partial\phi}{\partial x})^2\ -\ (\frac{\partial S}{\partial x})^2\ +\ \frac{1}{\phi^2}\ (\frac{\partial\phi}{\partial x})^2\ +\ 2\ \frac{i}{\phi}\ \frac{\partial S}{\partial x}\ \frac{\partial\phi}{\partial x}]\ \ \rightarrow$$

$$\frac{\partial^2\Psi}{\partial x^2}\ =\ e^{i\ S}\ [\frac{\partial^2\phi}{\partial x^2}\ +\ 2\ i\ \frac{\partial S}{\partial x}\ \frac{\partial\phi}{\partial x}\ +\ i\ \phi\ \frac{\partial^2 S}{\partial x^2}\ -\ \phi\ (\frac{\partial S}{\partial x})^2]\ =$$

$$=\ [\ i\ \frac{\partial^2 S}{\partial x^2}\ +\ \frac{1}{\phi}\ \frac{\partial^2\phi}{\partial x^2}\ -\ (\frac{\partial S}{\partial x})^2\ +\ 2\ i\ \frac{1}{\phi}\ \frac{\partial S}{\partial x}\ \frac{\partial\phi}{\partial x}\ ]\ \Psi\ . \qquad (2.3e,f)$$

Now, inserting the relations defined by eq. (2.3a,e) into eq. (2.1) we have, remembering that $e^{i\ S}$ is common factor:

$$i\ \hbar\ (\frac{\partial\phi}{\partial t}\ +\ i\ \phi\ \frac{\partial S}{\partial t})\ =\ -\ \frac{\hbar^2}{2\ m}\ [\frac{\partial^2\phi}{\partial x^2}\ +$$

$$+ 2\ i\ \frac{\partial S}{\partial x}\ \frac{\partial \phi}{\partial x}\ +\ i\ \phi\ \frac{\partial^2 S}{\partial x^2}\ -\ \phi\ (\frac{\partial S}{\partial x})^2]\ +\ [\ \frac{1}{2}\ m\ \Omega(t)^2\ x^2\ +\ \lambda\ x\ X(t)\ ]\ \phi\ -$$

$$-\ \frac{i\ \hbar}{4\ \tau}\ (\ \frac{[x\ -\ q(t)]^2}{\delta^2(t)}\ -\ 1\ )\ \phi\ , \qquad (2.4)$$

Separating the real and imaginary parts of the relation (2.4), results:

a) imaginary part

$$\frac{\partial \phi}{\partial t}\ =\ -\ \frac{\hbar}{2\ m}\ (\ 2\ \frac{\partial S}{\partial x}\ \frac{\partial \phi}{\partial x}\ +\ \phi\ \frac{\partial^2 S}{\partial x^2}\ )\ -\ \frac{1}{4\ \tau}\ (\ \frac{[x\ -\ q(t)]^2}{\delta^2(t)}\ -\ 1\ )\ \phi\ , \qquad (2.5)$$

b) real part

$$-\ \hbar\ \phi\ \frac{\partial S}{\partial t}\ =\ -\ \frac{\hbar^2}{2\ m}\ [\frac{\partial^2 \phi}{\partial x^2}\ -\ \phi\ (\frac{\partial S}{\partial x})^2]\ +\ [\ \frac{1}{2}\ m\ \Omega(t)^2\ x^2\ +\ \lambda\ x\ X(t)\ ]\ \phi \quad (\div\ m\ \phi) \quad \to$$

$$-\ \frac{\hbar}{m}\ \frac{\partial S}{\partial t}\ =\ -\ \frac{\hbar^2}{2\ m^2}\ \frac{1}{\phi}\ [\ \frac{\partial^2 \phi}{\partial x^2}\ -\ \phi\ (\frac{\partial S}{\partial x})^2\ ]\ +\ [\ \frac{1}{2}\ \Omega(t)^2\ x^2\ +\ \frac{\lambda}{m}\ x\ X(t)\ ]\ . \qquad (2.6)$$

Dynamics of the Schrödinger's Equation for Continuous Quantum Measurements

Now, let us see the correlation between the expressions (2.5-6) and the traditional equations of the Ideal Fluid Dynamics [5] a) continuity equation, b) Euler's equation. To do this let us perform the following correspondences:

Quantum density probability: $\mid \Psi(x,\ t)\mid^2\ =\ \Psi^*(x,\ t)\ \Psi(x,\ t) \quad \longleftrightarrow$

Quantum mass density: $\rho(x,\ t)\ =\ \phi^2(x,\ t) \quad \longleftrightarrow \quad \sqrt{\rho}\ =\ \phi\ , \qquad (2.7a,b)$

Gradient of the wave function: $\frac{\hbar}{m}\ \frac{\partial S(x,\ t)}{\partial x} \quad \longleftrightarrow$

Quantum velocity: $v_{qu}(x,\ t) \equiv v_{qu}\ . \qquad (2.8)$

Putting the relations (2.7b, 2.8) into the equation (2.5) we get:

$$\frac{\partial \sqrt{\rho}}{\partial t}\ =\ -\ \frac{\hbar}{2\ m}\ (\ 2\ \frac{\partial S}{\partial x}\ \frac{\partial \sqrt{\rho}}{\partial x}\ +\ \sqrt{\rho}\ \frac{\partial^2 S}{\partial x^2}\ )\ -\ \frac{1}{4\ \tau}\ (\ \frac{[x\ -\ q(t)]^2}{\delta^2(t)}\ -\ 1\ )\ \sqrt{\rho} \quad \to$$

$$\frac{1}{2\ \sqrt{\rho}}\ \frac{\partial \rho}{\partial t}\ =\ -\ \frac{\hbar}{2\ m}\ (\ 2\ \frac{\partial S}{\partial x}\ \frac{1}{2\ \sqrt{\rho}}\ \frac{\partial \rho}{\partial x}\ +\ \sqrt{\rho}\ \frac{\partial^2 S}{\partial x^2}\ )\ -\ \frac{1}{4\ \tau}\ (\ \frac{[x\ -\ q(t)]^2}{\delta^2(t)}\ -\ 1\ )\ \sqrt{\rho} \quad \to$$

$$\frac{1}{\rho}\ \frac{\partial \rho}{\partial t}\ =\ -\ \frac{\hbar}{m}\ (\ \frac{\partial S}{\partial x}\ \frac{1}{\rho}\ \frac{\partial \rho}{\partial x}\ +\ \frac{\partial^2 S}{\partial x^2}\ )\ -\ \frac{1}{2\ \tau}\ (\ \frac{[x\ -\ q(t)]^2}{\delta^2(t)}\ -\ 1\ ) \quad \to$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right) - \frac{1}{\rho} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right) \frac{\partial \rho}{\partial x} - \frac{1}{2\ \tau} \left(\frac{[x - q(t)]^2}{\delta^2(t)} - 1 \right) \rightarrow$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = - \frac{\partial v_{qu}}{\partial x} - \frac{v_{qu}}{\rho} \frac{\partial \rho}{\partial x} - \frac{1}{2\ \tau} \left(\frac{[x - q(t)]^2}{\delta^2(t)} - 1 \right) \rightarrow$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_{qu}}{\partial x} + v_{qu} \frac{\partial \rho}{\partial x} = - \frac{\rho}{2\ \tau} \left(\frac{[x - q(t)]^2}{\delta^2(t)} - 1 \right) \rightarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\ v_{qu})}{\partial x} = - \frac{\rho}{2\ \tau} \left(\frac{[x - q(t)]^2}{\delta^2(t)} - 1 \right) , \qquad (2.9)$$

which represents the continuity equation of the mass conservation law of the Fluid Dynamics. We must note that this expression also indicates descoerence of the considered physical system represented by the eq. (2.1). Using eq. (2.7b) we now define the quantum potential V_{qu}:

$$V_{qu}(x,\ t) \equiv V_{qu} = - \left(\frac{\hbar^2}{2\ m\ \phi} \right) \frac{\partial^2 \phi}{\partial x^2} = - \frac{\hbar^2}{2\ m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} , \qquad (2.10a,b)$$

the expression (2.6) will written as:

$$\frac{\hbar}{m} \frac{\partial S}{\partial t} + \frac{\hbar^2}{2\ m^2} \left(\frac{\partial S}{\partial x} \right)^2 = - \frac{1}{m} \Big(\left[\frac{1}{2}\ m\ \Omega^2(t)\ x^2 + \lambda\ x\ X(t) \right] + V_{qu} \Big) , \qquad (2.11)$$

or, using (2.8):

$$\hbar \frac{\partial S}{\partial t} + \frac{1}{2}\ m\ v_{qu}^2 + \frac{1}{2}\ m\ \Omega^2(t)\ x^2 + \lambda\ x\ X(t) + V_{qu} =$$

$$= \hbar \frac{\partial S}{\partial t} + \frac{1}{2}\ m\ v_{qu}^2 + V + V_{qu} = 0 . \qquad (2.12a,b)$$

Differentiating the relation (2.12a) with respect x and using the relation (2.8) we obtain:

$$\frac{\partial}{\partial x} \left[\hbar \frac{\partial S}{\partial t} + \Big(\frac{1}{2}\ m\ v_{qu}^2 + \left[\frac{1}{2}\ m\ \Omega^2(t)\ x^2 + \lambda\ x\ X(t) \right] + V_{qu} \Big) \right] = 0 \rightarrow$$

$$\frac{\partial}{\partial t} \left(\frac{\hbar}{m} \frac{\partial S}{\partial x} \right) + \frac{\partial}{\partial x} \left(\frac{1}{2}\ v_{qu}^2 \right) + \Omega^2(t)\ x + \frac{\lambda}{m}\ X(t) = - \frac{1}{m} \frac{\partial}{\partial x} V_{qu} \rightarrow$$

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \Omega^2(t)\ x + \frac{\lambda}{m}\ X(t) = - \frac{1}{m} \frac{\partial}{\partial x} V_{qu} , \qquad (2.13)$$

which is an equation similar to the Euler's equation which governs the motion of an ideal fluid.

Taking into account that[6]:

$$v_{qu}(x,\ t)\ =\ \frac{dx_{qu}}{dt}\ =\ [\ \frac{\dot{\delta}(t)}{\delta(t)}\ +\ \frac{1}{2\ \tau}\ ]\ [x_{qu}\ -\ q(t)]\ +\ \dot{q}(t)\ , \qquad (2.14\text{a,b})$$

the expression (2.13) could be written as:

$$m\ \frac{d^2x}{dt^2}\ =\ -\ \frac{\partial}{\partial x}\left[\ \frac{1}{2}\ m\ \Omega^2(t)\ x^2\ +\ \lambda\ x\ X(t)\ + V_{qu}\ \right] \equiv$$

$$\equiv F_c(x,\ t)\ |_{x=x(t)}\ +\ F_{qu}(x,\ t)\ |_{x=x(t)}\ , \qquad (2.15)$$

where:

$$\frac{d}{dt}\ =\ \frac{\partial}{\partial t}\ +\ v_{qu}\ \frac{\partial}{\partial x}\ , \qquad (2.16)$$

is the "substantive differentiation" (local plus convective) or "hidrodynamical differention" [5]. We note that the eq. (2.15) has a form of the second Newton law.

In this way, the expressions (2.10a,b;2.13;2.15) represent the dynamics of a quantum particle which propagates with the quantum velocity ($\vec{v}_{qu}$) in a viscous medium, submitted to a time dependent classical harmonic potential [$\frac{1}{2}\ m\ \Omega(t)^2\ x^2$], to an external linear field characterized by the potential [$\lambda\ x\ X(t)$] and to the quantum Bohm potential (V_{qu}).

In what follows we calculate the wave packet of the Schrödinger's Equation for Continuous Measurements ($SECM$) given by the eq.(2.1)

3. Quantum Wave Packet

3.1. Introduction

In 1909 [7], Einstein studied the black body radiation in thermodynamical equilibrium with matter. Starting from Planck's equation, of 1900, of the radiation density and using the Fourier expansion technique to calculate its fluctuations, he showed that it exhibits, simultaneously, fluctuations which are characteristic of waves and particles. In 1916 [8], analyzing again the black body Planckian radiation, Einstein proposed that an electromagnetic radiation with wavelenght λ had a linear momentum p, given by the relation:

$$\mathrm{p} = \frac{h}{\lambda}\ , \qquad (3.1.1)$$

where **h** is the Planck constant [9].

In works developed between 1923 and 1925 [10] de Broglie formulated his fundamental idea that the electron with mass m, in its atomic orbital motion with velocity v and linear momentum $p\ =\ m\ v$ is guided by a "matter wave" (pilot-wave) with wavelenght is given by:

$$\lambda\ =\ \frac{h}{p}\ . \qquad (3.1.2)$$

In 1926 [11], Schrödinger proposed that the "pilot-wave de Broglieаn" ought to obey a differential equation, today know as the famous Schrödinger's equation:

$$i \, \hbar \, \frac{\partial}{\partial t} \, \Psi(\vec{r}, \; t) = \; \hat{H} \; \Psi(\vec{r}, \; t) \; , \qquad (3.1.3a)$$

where $\hat{H}$ is Hamiltonian operator definied by:

$$\hat{H} \; = \; \frac{\hat{p}^2}{2 \; m} \; + \; V(\vec{r}, \; t) \; , \qquad (\hat{p} \; = \; - \; i \; \hbar \; \nabla) \; , \qquad (3.1.3b,c)$$

where V is the potential energy. In this same year of 1926 [12] Born interpreted the Schrödinger wave function Ψ as being an amplitude of probability.

3.2. Quantum Wave Packet via Schrödinger-Feynman Quantum Mechanics

As well known [13], when the potential energy V of physical system, with total energy E, depends only on the position $[V(\vec{r})]$, the solution of the Schrödinger's equation (SE) [see relations (3.1.3a-c)] is given by:

$$\Psi(\vec{r}, \; t) \; = \; \psi(\vec{r}) \; e^{- \; i \; \frac{E}{\hbar} \; t} \; = \; \psi(\vec{r}) \; e^{- \; i \; \omega \; t}, \qquad (3.2.1a,b)$$

where $\psi(\vec{r})$ satisfies the following equation, known as time independent SE [14]:

$$\Delta \; \psi(\vec{r}) \; + \; \frac{2 \; m}{\hbar^2} \; (E \; - \; V) \; \psi(\vec{r}) \; = \; 0 \quad \Leftrightarrow \quad \hat{H} \; \psi(\vec{r}) \; = \; E \; \psi(\vec{r}), \qquad (3.2.2a,b)$$

where $\hat{H}$ is given by the expressions (3.1.3b,c). In addition, $\psi(\vec{r})$ and its differentiation must be continuous $\frac{\partial \psi(\vec{r})}{\partial \vec{r}}$.

It is important to note that the relation (3.2.1b) was obtained considering the Planckian energy:

$$E \; = \; h \; \nu \; = \; \hbar \; \omega \; , \quad \hbar \; = \; \frac{h}{2 \; \pi}, \quad \omega \; = \; 2 \; \pi \; \nu \; . \qquad (3.2.3a\text{-}d)$$

As the expression (3.2.2b) is an eigenvalue equation its solution is given by a discrete set of eigenfunctions ("Schrödingerian waves") of the operator $\hat{H}$ [15]. On the other hand, the expression (3.1.2) suggests that it would be possible to use a handful concentration of "de Broglieаn waves", with wavelenght λ, to describe the particles localized in the space. In this description it is necessary to use a mechanism that takes into account these "waves" with many wavelenghts. This mechanism is the Fourier Analysis [15]. So, according to this technique (to one dimentional case) we can considerer ψ(x) like a superposition of plane monochromatic harmonic waves, that is:

$$\psi(x) \; = \; \frac{1}{\sqrt{2 \; \pi}} \int_{- \; \infty}^{+ \; \infty} \phi(k) \; e^{i \; k \; x} \; dk \; , \qquad (3.2.4a)$$

beeing:

$$\phi(k) \ = \ \frac{1}{\sqrt{2\ \pi}} \int_{-\ \infty}^{+\ \infty} \psi(x')\ e^{-\ i\ k\ x'}\ dx'\ , \qquad (3.2.4b)$$

where:

$$k \ = \ \frac{2\ \pi}{\lambda}\ , \qquad (3.2.4c)$$

is the wavenumber and represents the transition from the discrete to the continuous description. Note that, using the expressions (3.1.2), (3.2.3c), the relation (3.2.4c) will be written as:

$$k \ = \ 2\ \pi\ \frac{p}{h} \quad \rightarrow \quad p \ = \ \hbar\ k\ . \qquad (3.2.4d)$$

Inserting the expression (3.2.4a) in the same relation (3.2.4b) results:

$$\psi(x) \ = \ \frac{1}{2\ \pi} \int_{-\ \infty}^{+\ \infty} \int_{-\ \infty}^{+\ \infty} \psi(x')\ e^{i\ k\ (x-\ x')}\ dx'\ dk\ . \qquad (3.2.5)$$

Considering that [15]:

$$\delta(z'\ -\ z) \equiv \delta(z\ -\ z') \ = \ \frac{1}{2\ \pi} \int_{-\ \infty}^{+\ \infty} e^{i\ k\ (z-\ z')}\ dk\ , \qquad (3.2.6a,b)$$

$$f(z) \ = \ \int_{-\ \infty}^{+\ \infty} f(z')\ \delta(z'\ -\ z)\ dz'\ , \qquad (3.2.6c)$$

we verify that (since $z,\ z' \equiv x,\ x'$), the consistency of the relation (3.2.5), characterized by the famous completeness relation.

Taking into account the relation (3.2.4a) in the unidimensional representation of the equation (3.2.1b) the following relation will be obtained:

$$\Psi(x,\ t) \ = \ \frac{1}{\sqrt{2\ \pi}} \int_{-\ \infty}^{+\ \infty} \phi(k)\ e^{i\ [k\ x\ -\ \omega(k)\ t]}\ dk\ , \qquad (3.2.7)$$

which represents the wave packet of amplitude $\phi(k)$.

Note that the dependence ω in terms of k, indicated in the above relation, is due to the fact that the energy E of a physical system depends of p. So, considering that this fact and also the relations (3.2.3b;3.2.4d), this dependence is verified immediately as will be shown in what follows.

Now, let us write the equation (3.2.7) in terms of the Feynman propagator. Putting $t \ = \ 0$ in this expression and, in analogy with the relation (3.2.4b), we get:

$$\Psi(x,\ 0) \ = \ \frac{1}{\sqrt{2\ \pi}} \int_{-\ \infty}^{+\ \infty} \phi(k)\ e^{i\ k\ x}\ dk \quad \rightarrow$$

$$\phi(k) \ = \ \frac{1}{\sqrt{2\ \pi}} \int_{-\ \infty}^{+\ \infty} \Psi(x',\ 0)\ e^{-\ i\ k\ x'}\ dx'\ . \qquad (3.2.8)$$

Inserting this expression in the equation (3.2.7) results:

$$\Psi(x,\ t) \ = \ \frac{1}{\sqrt{2\ \pi}} \int_{-\ \infty}^{+\ \infty} \Big(\frac{1}{\sqrt{2\ \pi}} \int_{-\ \infty}^{+\ \infty} \Psi(x',\ 0)\ e^{-\ i\ k\ x'}\ dx' \Big) \times e^{i\ [k\ x\ -\ \omega(k)\ t]}\ dk \ \ \rightarrow$$

$$\Psi(x,\ t) \ = \ \int_{-\ \infty}^{+\ \infty} \Big(\frac{1}{2\ \pi} \int_{-\ \infty}^{+\ \infty} e^{i\ k\ [(x\ -\ x')\ -\ \frac{\omega(k)}{k}\ t]}\ dk \Big) \times \Psi(x',\ 0)\ dx'\ . \qquad (3.2.9)$$

At this point it is important to say that, in the formalism of Feynman Quantum Mechanics [16], the term inside the brackets in the equation (3.2.9) represents the Feynman propagator $K(x,\ x';\ t)$. So, this expression can be written as:

$$\Psi(x,\ t) \ = \ \int_{-\ \infty}^{+\ \infty} K(x,\ x';\ t)\ \Psi(x',\ 0)\ dx'\ , \qquad (3.2.10a)$$

where:

$$K(x,\ x';\ t) \ = \ \frac{1}{2\ \pi} \int_{-\ \infty}^{+\ \infty} e^{i\ k\ [(x\ -\ x')\ -\ \frac{\omega(k)}{k}\ t]}\ dk\ . \qquad (3.2.10b)$$

The equation (3.2.10a) represents the wavefunction Ψ for any time t in terms of this function in the time $t \ = \ 0$. So, if ω(k) is a known function of k, so $\Psi(x,\ t)$ can be explicitely obtained from $\Psi(x,\ 0)$.

In the sequence we determine the form of the wave packet given in the equations (3.2.10a,b) according to the $SECM$, defined by eq.(2.1) [17].

3.3. The Quantum Wave Packet of the Schrödinger for Continuous Measurements

Initially, let us calculate the quantum trajectory (x_{qu}) of the physical system represented by the eq.(2.1). To do this, let us integrate the relations (2.14b) [remembering that $\int \frac{dz}{z} \ = \ \ell n\ z,\ \ell n\ (\frac{x}{y}) \ = \ \ell n\ x \ - \ \ell n\ y$,e $\ell n\ x\ y \ = \ \ell n\ x \ + \ \ell n\ y$]:

$$v_{qu}(x,\ t) \ = \ \frac{dx_{qu}}{dt} \ = \ [\ \frac{\dot{\delta}(t)}{\delta(t)} \ + \ \frac{1}{2\ \tau}\ ]\ [x_{qu} \ - \ q(t)] \ + \ \dot{q}(t) \ \ \rightarrow$$

$$\frac{dx_{qu}}{dt} \ - \ \frac{dq}{dt} \ = \ [\ \frac{\dot{\delta}(t)}{\delta(t)} \ + \ \frac{1}{2\ \tau}\ ]\ [x_{qu} \ - \ q(t)] \ \ \rightarrow \ \ \frac{d[x_{qu}(t) \ - \ q(t)]}{[x_{qu}(t) \ - \ q(t)]} \ = \ [\ \frac{\dot{\delta}(t)}{\delta(t)}\ dt \ + \ \frac{dt}{2\ \tau}\ ] \ \ \rightarrow$$

$$\int_{o}^{t} \frac{d[x_{qu}(t') \ - \ q(t')]}{[x_{qu}(t') \ - \ q(t')]} \ = \ \int_{o}^{t} \frac{d\delta(t')}{\delta(t')} \ + \ \int_{o}^{t} \frac{dt}{2\ \tau} \ \ \rightarrow$$

$$\ell n\ \Big(\frac{[x_{qu}(t) \ - \ q(t)]}{[x_{qu}(0) \ - \ q(0)]} \Big) \ = \ \ell n\ \Big[\frac{\delta(t)}{\delta(0)} \Big] \ + \ \frac{t}{2\ \tau} \ = \ \ell n\ \Big[\frac{\delta(t)}{\delta(0)} \Big] \ + \ \ell n\ \Big[exp\ (\ \frac{t}{2\ \tau}\ ) \Big] \ =$$

$$= \ \ell n\ \Big(\frac{\delta(t)}{\delta(0)}\ .\ exp\ \Big[\frac{t}{2\ \tau} \Big] \Big) \ \ \rightarrow \ \ x_{qu}(t) \ = \ q(t) \ + \ e^{t/2\ \tau}\ \frac{\delta(t)}{\delta(0)}\ [x_{qu}(0) \ - \ q(0)]\ , \qquad (3.3.1)$$

that represent the looked for quantum trajectory.

To obtain the Schrödinger-de Broglie-Bohm wave packet for Continuous Measurements given by the eq.(2.2), let us expand the functions $S(x,\ t)$, $V(x,\ t)$ and $V_{qu}(x,\ t)$ around of $q(t)$ up to second Taylor order [2.5]. In this way we have:

$$S(x,\ t)\ =\ S[q(t),\ t]\ +\ S'[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{S''[q(t),\ t]}{2}\ [x\ -\ q(t)]^2\ , \qquad (3.3.2)$$

$$V(x,\ t)\ =\ V[q(t),\ t]\ +\ V'[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{V''[q(t),\ t]}{2}\ [x\ -\ q(t)]^2\ , \qquad (3.3.3)$$

$$V_{qu}(x,\ t)\ =\ V_{qu}[q(t),\ t]\ +\ V'_{qu}[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{V''_{qu}[q(t),\ t]}{2}\ [x\ -\ q(t)]^2\ . \qquad (3.3.4)$$

Differentiating the expression (3.3.2) in the variable x, multiplying the result by $\frac{\hbar}{m}$, using the relations (2.8) and (2.14b), and taking into account the polynomial identity property, we obtain:

$$\frac{\hbar}{m}\ \frac{\partial S(x,\ t)}{\partial x}\ =\ \frac{\hbar}{m}\ \Big(\ S'[q(t),\ t]\ +\ S''[q(t),\ t]\ [x\ -\ q(t)]\ \Big)\ =$$

$$=\ v_{qu}(x,\ t)\ =\ [\ \frac{\dot{\delta}(t)}{\delta(t)}\ +\ \frac{1}{2\ \tau}\ ]\ [x_{qu}\ -\ q(t)]\ +\ \dot{q}(t)\ =\ \ \rightarrow$$

$$S'[q(t),\ t]\ =\ \frac{m\ \dot{q}(t)}{\hbar}\ ,\quad S''[q(t),\ t]\ =\ \frac{m}{\hbar}\ [\ \frac{\dot{\delta}(t)}{\delta(t)}\ +\ \frac{1}{2\ \tau}\ ]\ . \qquad (3.3.5a,b)$$

Substituting the expressions (3.3.5a,b) in the equation (3.3.2), results:

$$S(x,\ t)\ =\ S_o(t)\ +\ \frac{m\ \dot{q}(t)}{\hbar}\ [x\ -\ q(t)]\ +\ \frac{m}{2\ \hbar}\ \Big[\ \frac{\dot{\delta}(t)}{\delta(t)}\ +\ \frac{1}{2\ \tau}\ \Big]\ [x\ -\ q(t)]^2\ , \qquad (3.3.6)$$

where:

$$S_o(t) \equiv S[q(t),\ t]\ , \qquad (3.3.7)$$

is the quantum action.

Differentiating the eq.(3.3.6) in relation to the time t, we obtain (remembering that $\frac{\partial x}{\partial t} = 0$):

$$\frac{\partial S}{\partial t}\ =\ \dot{S}_o(t)\ +\ \frac{\partial}{\partial t}\ \Big(\ \frac{m\ \dot{q}(t)}{\hbar}\ [x\ -\ q(t)]\ \Big)\ +\ \frac{\partial}{\partial t}\ \left(\ \frac{m}{2\ \hbar}\ \Big[\ \frac{\dot{\delta}(t)}{\delta(t)}\ +\ \frac{1}{2\ \tau}\ \Big]\ [x\ -\ q(t)]^2\ \right)\ \ \rightarrow$$

$$\frac{\partial S}{\partial t}\ =\ \dot{S}_o(t)\ +\ \frac{m\ \ddot{q}(t)}{\hbar}\ [x\ -\ q(t)]\ -\ \frac{m\ \dot{q}(t)^2}{\hbar}\ +$$

$$+\ \frac{m}{2\ \hbar}\ [\frac{\ddot{\delta}(t)}{\delta(t)}\ -\ \frac{\dot{\delta}^2(t)}{\delta^2(t)}]\ [x\ -\ q(t)]^2\ -\ \frac{m\ \dot{q}(t)}{\hbar}\ \Big(\ \frac{\dot{\delta}(t)}{\delta(t)}\ +\ \frac{1}{2\ \tau}\ \Big)\ [x\ -\ q(t)]\ . \qquad (3.3.8)$$

Considering that [6]:

$$\rho(x,\ t) \ = \ [2\ \pi\ \delta^2(t)]^{-\ 1/2}\ e^{-\ \frac{[x\ -\ \bar{x}(t)]^2}{2\ \delta^2(t)}}\ , \qquad (3.3.9)$$

let us write V_{qu} in terms of $[x\ -\ q(t)]$. Initially using the eqs.(2.7b) and (3.3.9), we calculate the following differentiations:

$$\frac{\partial\phi}{\partial x} \ = \ \frac{\partial}{\partial x}\Big(\ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \Big) \ = \ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \frac{\partial}{\partial x}\Big(\ -\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}\ \Big) \ \rightarrow$$

$$\frac{\partial\phi}{\partial x} \ = \ -\ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \frac{[x\ -\ q(t)]}{2\ \delta^2(t)}\ ,$$

$$\frac{\partial^2\phi}{\partial x^2} \ = \ \frac{\partial}{\partial x}\Big(\ -\ \ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \frac{[x\ -\ q(t)]}{2\ \delta^2(t)}\ \Big) =$$

$$= \ -\ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \frac{\partial}{\partial x}\Big(\ \frac{[x\ -\ q(t)]}{2\ \delta^2(t)}\ \Big)\ -$$

$$-\ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \frac{\partial}{\partial x}\Big(\ -\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}\ \Big)\ (\ \frac{[x\ -\ q(t)]}{2\ \delta^2(t)}\ )\ \ \rightarrow$$

$$\frac{\partial^2\phi}{\partial x^2} \ = \ -\ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \frac{1}{2\ \delta^2(t)}\ +\ [2\ \pi\ \delta^2(t)]^{-\ 1/4}\ e^{-\ \frac{[x\ -\ q(t)]^2}{4\ \delta^2(t)}}\ \frac{[x\ -\ q(t)]^2}{4\ \delta^4(t)}\ =$$

$$= \ -\ \phi\ \frac{1}{2\ \delta^2(t)}\ +\ \phi\ \frac{[x\ -\ q(t)]^2}{4\ \delta^4(t)}\ \ \rightarrow\ \ \frac{1}{\phi}\ \frac{\partial^2\phi}{\partial x^2}\ =\ \frac{[x\ -\ q(t)]^2}{4\ \delta^4(t)}\ -\ \frac{1}{2\ \delta^2(t)}\ . \qquad (3.3.10)$$

Substituting the relation (3.3.10) in the equation (2.10a), taking into account the expression (3.3.4), results:

$$V_{qu}(x,\ t) \ = \ V_{qu}[q(t),\ t]\ +\ V'_{qu}[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{V''_{qu}[q(t),\ t]}{2}\ [x\ -\ q(t)]^2 \ \ \rightarrow$$

$$V_{qu}(x,\ t) \ = \ \frac{\hbar^2}{4\ m\ \delta^2(t)}\ [x\ -\ q(t)]^o\ -\ \frac{\hbar^2}{8\ m\ \delta^4(t)}\ [x\ -\ q(t)]^2\ . \qquad (3.3.11)$$

Besides this the eq.(3.3.3) will be written, using the eq.(2.1) in the form:

$$V(x,\ t) \ = \ V[q(t),\ t]\ +\ V'[q(t),\ t]\ [x\ -\ q(t)]\ +\ \frac{V''[q(t),\ t]}{2}\ [x\ -\ q(t)]^2 \ \ \rightarrow$$

$$V(x,\ t) \ = \ \frac{1}{2}\ m\ \Omega^2(t)\ q^2(t)\ +\ \lambda\ q(t)\ X(t)\ +$$

$$+\ \Big(\ m\ \Omega^2(t)\ q(t)\ +\ \lambda\ X(t)\ \Big)\ [x\ -\ q(t)]\ +\ \frac{m}{2}\ \Omega^2(t)\ [x\ -\ q(t)]^2\ . \qquad (3.3.12)$$

Inserting the relations (2.8), (2.14b) and (3.3.2-4;3.3.8,10,11), into the eq.(2.12b), we obtain, remembering that $S_o(t)$, $\delta(t)$ and $q(t)$:

$$\hbar \frac{\partial S}{\partial t} + \frac{1}{2} m v_{qu}^2 + V + V_{qu} =$$

$$= \hbar \Big[\dot{S}_o + \frac{m \ddot{q}}{\hbar} (x - q) - \frac{m \dot{q}^2}{\hbar} + \frac{m}{2 \hbar} (\frac{\ddot{\delta}}{\delta} - \frac{\dot{\delta}^2}{\delta^2}) (x - q)^2 -$$

$$- \frac{m \dot{q}}{\hbar} (\frac{\dot{\delta}}{\delta} + \frac{1}{2 \tau}) (x - q) \Big] + \frac{1}{2} m \Big[(\frac{\dot{\delta}}{\delta} + \frac{1}{2 \tau}) (x - q) + \dot{q} \Big]^2 +$$

$$+ \frac{1}{2} m \Omega^2(t) q^2 + \lambda q X(t) + [m \Omega^2(t) q + \lambda X(t)] (x - q) + \frac{m}{2} \Omega^2(t) (x - q)^2 +$$

$$+ \frac{\hbar^2}{4 m \delta^2} - \frac{\hbar^2}{8 m \delta^4} (x - q)^2 = 0 . \qquad (3.3.13)$$

Since $(x - q)^o = 1$, we can gather together the above expression in potencies of $(x - q)$, obtaining:

$$\Big[\hbar \dot{S}_o - m \dot{q}^2 + \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \Omega^2(t) q^2 + \lambda q X(t) + \frac{\hbar^2}{4 m \delta^2} \Big] (x - q)^o +$$

$$+ \Big[m \ddot{q} - m \dot{q} (\frac{\dot{\delta}}{\delta} + \frac{1}{2 \tau}) + m \dot{q} (\frac{\dot{\delta}}{\delta} + \frac{1}{2 \tau}) + m \Omega^2(t) q + \lambda X(t) \Big] (x - q) + \Big[\frac{m}{2} (\frac{\ddot{\delta}}{\delta} - \frac{\dot{\delta}^2}{\delta^2}) +$$

$$+ \frac{m}{2} (\frac{\dot{\delta}^2}{\delta^2} + \frac{\dot{\delta}}{\tau \delta} + \frac{1}{4 \tau^2}) + \frac{m}{2} \Omega^2(t) - \frac{\hbar^2}{8 m \delta^4} \Big] (x - q)^2 = 0 . \qquad (3.3.14)$$

As the above relation is an identically null polynomium, the coefficients of the potencies must be all equal to zero, that is:

$$\dot{S}_o(t) = \frac{1}{\hbar} \Big[\frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \Omega^2(t) q^2 - \lambda q X(t) - \frac{\hbar^2}{4 m \delta^2} \Big] , \qquad (3.3.15)$$

$$\ddot{q} + \Omega^2(t) q + \frac{\lambda}{m} X(t) = 0 , \qquad (3.3.16)$$

$$\ddot{\delta} + \frac{\dot{\delta}}{\tau} + \Big[\Omega^2(t) + \frac{1}{4 \tau^2} \Big] \delta = \frac{\hbar^2}{4 m^2 \delta^3(t)} . \qquad (3.3.17)$$

Assuming that the following initial conditions are obeyed:

$$q(0) = x_o , \quad \dot{q}(0) = v_o , \quad \delta(0) = a_o , \quad \dot{\delta}(0) = b_o , \qquad (3.3.18\text{a-d})$$

and that [see eq.(3.3.7)]:

$$S_o(0) = \frac{m v_o x_o}{\hbar} , \qquad (3.3.19)$$

the integration of the expression (3.3.15) will be given by:

$$S_o(t) \ = \ \frac{1}{\hbar} \int_o^t dt' \left[\ \frac{1}{2} \ m \ \dot{q}^2(t') \ - \ \frac{1}{2} \ m \ \Omega^2(t') \ q^2(t') - \right.$$

$$\left. - \ \lambda \ q(t') \ X(t') \ - \ \frac{\hbar^2}{4 \ m \ \delta^2(t')} \ \right] \ + \ \frac{m \ v_o \ x_o}{\hbar} \ . \qquad (3.3.20)$$

Taking into account the expressions (3.3.5a,b) and (3.3.20) in the equation (3.3.6) results:

$$S(x, \ t) \ = \ \frac{1}{\hbar} \int_o^t dt' \left[\ \frac{1}{2} \ m \ \dot{q}^2(t') \ - \ \frac{1}{2} \ m \ \Omega^2(t') \ q^2(t') \ - \ \lambda \ q(t') \ X(t') \ - \ \frac{\hbar^2}{4 \ m \ \delta^2(t')} \ \right] +$$

$$+ \ \frac{m \ v_o \ x_o}{\hbar} \ + \ \frac{m \ \dot{q}(t)}{\hbar} \ [x \ - \ q(t)] \ + \ \frac{m}{2 \ \hbar} \left[\ \frac{\dot{\delta}(t)}{\delta(t)} \ + \ \frac{1}{2 \ \tau} \ \right] [x \ - \ q(t)]^2 \ . \qquad (3.3.21)$$

This result obtained above permit us, finally, to obtain the wave packet for the *SBBMC* equation. Indeed, considering the relations (2.2;2.7b), (3.3.9) and (3.3.21), we get [18]:

$$\Psi(x, \ t) \ = \ [2 \ \pi \ \delta^2(t)]^{- \ 1/4} \ exp \left[\left(\ \frac{i \ m}{2 \ \hbar} \left[\ \frac{\dot{\delta}(t)}{\delta(t)} \ + \ \frac{1}{2 \ \tau} \ \right] \ - \ \frac{1}{4 \ \delta^2(t)} \ \right) [x \ - \ q(t)]^2 \ \right] \times$$

$$\times \ exp \left[\ \frac{i \ m \ \dot{q}(t)}{\hbar} \ [x \ - \ q(t)] \ + \ \frac{i \ m \ v_o \ x_o}{\hbar} \ \right] \times$$

$$\times \ exp \left[\ \frac{i}{\hbar} \int_o^t dt' \left[\ \frac{1}{2} \ m \ \dot{q}^2(t') \ - \ \frac{1}{2} \ m \ \Omega^2(t') \ q^2(t') - \ \lambda \ q(t') \ X(t') \ - \ \frac{\hbar^2}{4 \ m \ \delta^2(t')} \ \right] \right. . \qquad (3.3.22)$$

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